

A new method of randomization of lattice rules for multiple integration

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Abstract

Cranley and Patterson put forward the following randomization as the basis for the estimation of the error of a lattice rule for an integral of a one-periodic function over the unit cube in s dimensions. The lattice rule is randomized using independent random shifts in each coordinate direction that are uniformly distributed in the interval $[0, 1]$. This randomized lattice rule results in an unbiased estimator of the multiple integral. However, in practice, random variables that are independent and uniformly distributed on $[0, 1]$ are not available, since this would require an infinite number of random independent bits. A more realistic practical implementation of the Cranley and Patterson randomization uses rs independent random bits, in the following way. The lattice rule is randomized using independent random shifts in each coordinate direction that are uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$, where r may be large. For a rank-1 lattice rule with 2^m quadrature points and $r \geq m$, we show that this randomized lattice rule leads to an estimator of the multiple integral that typically has a large bias. We therefore propose that these rs independent random bits be used to perform a new randomization that employs an extension, in the number of quadrature points, to a lattice rule with 2^{m+sr} quadrature points (leading to embedded lattice rules). This new randomization is shown to lead to an estimator of the multiple integral that has much smaller bias.

Keywords: Error estimation; Extended lattice rule; Lattice rule; Randomization.

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1. Introduction

In this paper we consider the problem of computing

$$If = \int_{C^s} f(\mathbf{x}) d\mathbf{x},$$

where $f : \mathbb{R}^s \rightarrow \mathbb{R}$, $C^s = [0, 1]^s$ and the function f is one-periodic with respect to each component of \mathbf{x} , i.e. $f(\mathbf{x}) = f(\mathbf{x} + \mathbf{z})$ for all $\mathbf{z} \in \mathbb{Z}^s$ and $\mathbf{x} \in \mathbb{R}^s$. We suppose that f has an absolutely convergent Fourier series representation

$$f(\mathbf{x}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \hat{f}(\mathbf{h}) e^{2\pi i \mathbf{h} \cdot \mathbf{x}},$$

where $\mathbf{h} \cdot \mathbf{x} = h_1 x_1 + \dots + h_s x_s$ is the inner product in \mathbb{R}^s . We suppose further that f is known to belong to some class of functions F of smooth functions, with smoothness measured by the rate of decay of the Fourier coefficients. For classes of sufficiently smooth functions, a remarkably accurate approximation to If is provided by a lattice rule

$$Qf = \frac{1}{N} \sum_{j=0}^{N-1} f(\mathbf{x}_j),$$

where $\{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\}$ are the points of a carefully chosen integration lattice $L \subset \mathbb{R}^s$ that lie in the half-open cube $[0, 1)^s$. An integration lattice in \mathbb{R}^s is defined as a discrete subset of \mathbb{R}^s which is closed under addition and subtraction and which contains \mathbb{Z}^s as a subset. A very readable introduction to lattice rules is provided by Sloan and Joe (1994).

The standard method for estimating the lattice rule error $Qf - If$ is the randomization method due to Cranley and Patterson (1976). Define the shifted lattice rule

$$Q_c f = \frac{1}{N} \sum_{j=0}^{N-1} f(\{\mathbf{x}_j + \mathbf{c}\})$$

where $\mathbf{c} \in \mathbb{R}^s$ and $\{\mathbf{x}\}$ denotes the fractional part of the vector \mathbf{x} , obtained by taking the fractional part of each component of \mathbf{x} . Theorem 2.10 of Sloan and Joe (1994) states that

$$Q_c f - If = \sum'_{\mathbf{h} \in L^\perp} e^{2\pi i \mathbf{h} \cdot \mathbf{c}} \hat{f}(\mathbf{h}), \quad (1)$$

where the prime on the sum indicated that the zero term is omitted from the sum and L^\perp denotes the dual lattice, defined e.g. on p.26 of Sloan and Joe (1994).

Let $y_1 = Q_{\mathbf{u}_1} f, \dots, y_q = Q_{\mathbf{u}_q} f$, where $\mathbf{u}_1, \dots, \mathbf{u}_q$ are independent and identically distributed (iid) uniformly in the unit cube $[0, 1]^s$. Note that \mathbf{u} being uniformly

distributed in $[0, 1]^s$ is equivalent to the components of \mathbf{u} being iid uniformly distributed on $[0, 1]$. Cranley and Patterson (1976) propose that If be estimated by

$$\bar{y} = \frac{1}{q} \sum_{k=1}^q y_k.$$

As is well-known, $E(y_k) = If$ for $k = 1, \dots, q$, so that $E(\bar{y}) = If$. In other words, y_k is an unbiased estimator of If for $k = 1, \dots, q$, so that \bar{y} is an unbiased estimator of If . An expression for $\text{Var}(\bar{y})$, in terms of the Fourier coefficients of f , is provided by Proposition 4 of L'Ecuyer and Lumieux (2000). We review these known results in the Appendix and show that expressions for higher-order moments can also be found in terms of the Fourier coefficients of f .

The theory for the randomization proposed by Cranley and Patterson (1976) is elegant and relatively simple. However, this form of randomization is an idealisation. In practice, observations of random variables that are independent and uniformly distributed on $[0, 1]$ are not available. Henceforth, when we refer to iid random bits b_1, b_2, \dots , we assume that $P(b_1 = 0) = P(b_1 = 1) = 1/2$. Even a single random variable uniformly distributed on $[0, 1]$, is equivalent to an infinite number of iid random bits b_1, b_2, \dots . Specifically, if u is uniformly distributed on $[0, 1]$ then $u = .b_1b_2\dots$.

If one replaces these independent and uniformly distributed (on $[0, 1]$) random variables by pseudorandom numbers then the resulting estimator of the multiple integral will have largely unpredictable properties. What properties might be rigorously established would be hard won and of limited scope and would come nowhere near the elegance and simplicity of the theory for the randomization put forward by Cranley and Patterson (1976).

Observations of truly random bits may be taken from physical sources such as electronic thermal noise and various sources that rely on quantum-mechanical effects. Such observations may be obtained through the internet from reputable sources such as the *ANU Quantum Random Numbers Server* (<http://qrng.anu.edu.au>). Of course, in practice, one can obtain observations of only a finite number of random bits.

A realistic practical implementation of the Cranley and Patterson randomization uses rs independent random bits, in the following way. The lattice rule is randomized using independent random shifts in each coordinate direction that are uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$, where r may be large.

For simplicity of exposition, we consider the particular case that the integration lattice is generated by a rank-1 lattice rule

$$Qf = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\left\{\frac{j}{N}\mathbf{z}\right\}\right), \quad (2)$$

where the number of quadrature points $N = 2^m$, $\mathbf{z} \in \mathbb{Z}^s$ and has no common factors with N . We call \mathbf{z} the *generating vector*. For the definition of the rank of a lattice rule see e.g. Sloan and Joe (1994). Rank-1 lattice rules were introduced and extensively analysed by Korobov (1959). These rules may be found using the component-by-component (CBC) construction (see e.g. Dick, Kuo and Sloan, 2014). Also, for simplicity, we suppose that $r \geq m$. In Section 2, we show that this randomization leads to an estimator of the multiple integral that typically has a large bias. Note that s random variables that are iid and uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$ can be transformed to rs iid random bits b_1, \dots, b_{rs} and vice versa.

We therefore propose, in Section 3, that these rs iid random bits be used to perform a new randomization that employs an extension, in the number of quadrature points, from a rank-1 lattice rule with 2^m quadrature points to a rank-1 lattice rule with 2^{m+sr} quadrature points. This new randomization is shown to lead to an estimator of the multiple integral If that has much smaller bias. Some numerical illustrations of this property are provided in Section 4.

2. Results for randomization (of the type proposed by Cranley and Patterson, 1976) using iid random variables uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$

Consider the rank-1 lattice rule (2), where $N = 2^m$. The randomization of this lattice rule proposed by Cranley and Patterson (1976) is

$$Q_{\mathbf{u}}f = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\left\{\frac{j}{N}\mathbf{z} + \mathbf{u}\right\}\right), \quad (3)$$

where the random vector \mathbf{u} is uniformly distributed on $[0, 1]^s$. As already noted in the introduction, it is unrealistic to assume that we have observations of the infinite number of truly random bits that are required to produce an observation of \mathbf{u} . The randomized lattice rule that corresponds to (3) and is based on only a finite number rs of random bits is

$$Q_{\mathbf{v}}f = \frac{1}{N} \sum_{j=0}^{N-1} f\left(\left\{\frac{j}{N}\mathbf{z} + \mathbf{v}\right\}\right),$$

where the random vector \mathbf{v} is uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}^s$, where r may be large. For simplicity, we assume that $r \geq m$. In this section, we show that the bias $E(Q_{\mathbf{v}}f) - If$ can be unacceptably large for f in the class F .

To find $E(Q_{\mathbf{v}}f)$, we proceed as follows. The k 'th component of the random s -vector

$$\left\{ \frac{j}{N} \mathbf{z} + \mathbf{v} \right\} = \left\{ \frac{j}{2^m} \mathbf{z} + \mathbf{v} \right\} \quad (4)$$

is

$$\left\{ \frac{j}{2^m} z_k + v_k \right\}, \quad (5)$$

where z_k and v_k denote the k 'th components of \mathbf{z} and \mathbf{v} , respectively. Since we have assumed that $r \geq m$, (5) is uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$. Also, the components of the random s -vector (4) are independent random variables since the components of $\mathbf{v} = (v_1, \dots, v_s)$ are independent. Thus (4) has the same probability distribution as \mathbf{v} .

Hence

$$\begin{aligned} E(Q_{\mathbf{v}}f) &= E \left[\frac{1}{N} \sum_{j=0}^{N-1} f \left(\left\{ \frac{j}{N} \mathbf{z} + \mathbf{v} \right\} \right) \right] \\ &= \frac{1}{N} \sum_{j=0}^{N-1} E \left[f \left(\left\{ \frac{j}{N} \mathbf{z} + \mathbf{v} \right\} \right) \right] \\ &= \frac{1}{N} \sum_{j=0}^{N-1} E[f(\mathbf{v})] \\ &= E[f(\mathbf{v})] \\ &= \frac{1}{2^{sr}} \sum_{j_1=0}^{2^r-1} \cdots \sum_{j_s=0}^{2^r-1} f \left(\left(\frac{j_1}{2^r}, \dots, \frac{j_s}{2^r} \right) \right). \end{aligned}$$

This is just the product-rectangle rule with 2^{sr} quadrature points. This rule is known to be a particularly bad lattice rule for the types of function classes F under consideration. Furthermore, the number of quadrature points is only 2^{sr} , functionally independent of m . Therefore, the magnitude of the bias $|E(Q_{\mathbf{v}}f) - If|$ will take unacceptably large values for some f 's in the class of functions F . This is a very serious disadvantage of this form of randomization.

3. New randomization using iid random variables uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$ and two embedded lattice rules

Note that s random variables that are iid and uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$ can be transformed to rs iid random bits b_1, \dots, b_{rs} and vice versa. In this section, we describe a new randomization that uses these random bits to randomize the rank-1 lattice rule (2), which has $N = 2^m$ quadrature points. This randomization has far better properties than the randomization described in the previous section that uses the same number of iid random bits.

We choose the generating vector \mathbf{z} such that both the rank-1 lattice rule

$$Qf = \frac{1}{2^m} \sum_{j=0}^{2^m-1} f\left(\left\{\frac{j}{2^m}\mathbf{z}\right\}\right) \quad (6)$$

and the rank-1 lattice rule

$$\tilde{Q}f = \frac{1}{2^{m+sr}} \sum_{k=0}^{2^{m+sr}-1} f\left(\left\{\frac{k}{2^{m+sr}}\mathbf{z}\right\}\right) \quad (7)$$

perform well in the class F of functions f . This generating vector can be found using the method of Cools, Kuo and Nuyens (2006). The quadrature points of Qf are embedded in the set of quadrature points of $\tilde{Q}f$. Extensions of lattice rules in both dimension s and number of quadrature points were introduced by Hickernell, Hong, L'Ecuyer and Lemieux (2000). An existence proof for good extensible rank-1 lattice rules is provided by Hickernell and Niederreiter (2003).

Let w be a random variable that is uniformly distributed on $\{0, 1/2^{sr}, \dots, (2^{sr} - 1)/2^{sr}\}$. We can express w in binary form as $w = .b_1 \dots b_{sr}$, where b_1, \dots, b_{sr} are iid random bits. The new randomized rank-1 lattice rule is

$$Q'_w f = \frac{1}{2^m} \sum_{j=0}^{2^m-1} f\left(\left\{\frac{j+w}{2^m}\mathbf{z}\right\}\right)$$

It may be shown that $E[Q'_w f] = \tilde{Q}f$ as follows.

$$\begin{aligned}
E[Q'_w f] &= \frac{1}{2^m} \sum_{j=0}^{2^m-1} E \left[f \left(\left\{ \frac{j+w}{2^m} \mathbf{z} \right\} \right) \right] \\
&= \frac{1}{2^m} \sum_{j=0}^{2^m-1} \frac{1}{2^{sr}} \sum_{\ell=0}^{2^{sr}-1} f \left(\left\{ \frac{j + (\ell/2^{sr})}{2^m} \mathbf{z} \right\} \right) \\
&= \frac{1}{2^{m+sr}} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{2^{sr}-1} f \left(\left\{ \frac{2^{sr}j + \ell}{2^{m+sr}} \mathbf{z} \right\} \right) \\
&= \frac{1}{2^{m+sr}} \sum_{k=0}^{2^{m+sr}-1} f \left(\left\{ \frac{k}{2^{m+sr}} \mathbf{z} \right\} \right),
\end{aligned}$$

since $2^{sr}j + \ell$ can be expressed as a binary number using $m + sr$ bits.

Let $y_1 = Q'_{w_1} f, \dots, y_q = Q'_{w_q} f$, where w_1, \dots, w_q are iid uniformly distributed on $\{0, 1/2^{sr}, \dots, (2^{sr} - 1)/2^{sr}\}$. We estimate If by

$$\bar{y} = \frac{1}{q} \sum_{k=1}^q y_k.$$

Obviously, $E(\bar{y}) = \tilde{Q}f$. Therefore, \bar{y} is a biased estimator of If . However, the way in which the generating vector \mathbf{z} has been chosen implies that the bias $E(\bar{y}) - If = \tilde{Q}f - If$ will be small.

So far, we have randomized using iid random bits because this is the way that observations of truly random variables usually present themselves in practice. We remark that it is straightforward to develop corresponding results for randomizations using iid random variables that are uniformly distributed on $\{0, \dots, b-1\}$ for some base b other than $b = 2$.

4. Numerical illustrations

To numerically illustrate the much smaller bias of the estimator that results from the new randomization described in the previous section, we consider the numerical integration of a function f whose integral If takes a known value. A convenient class of such functions, which has been used extensively in the construction of lattice rules with good properties, is described on p. 72–73 of Sloan and Joe (1994). The particular member of this of this class that we consider is the following. For $\mathbf{x} = (x_1, \dots, x_s)$, we suppose that

$$f(\mathbf{x}) = \prod_{i=1}^s (1 + B_2(x_i)),$$

where $B_2(x) = x^2 - x + (1/6)$. Obviously, $If = 1$.

We consider rank-lattice rules, of the form first proposed by Korobov, for which the generating vector

$$\mathbf{z} = (1, \ell, \ell^2, \dots, \ell^{s-1}),$$

where ℓ is a carefully chosen positive integer. In particular, we consider the following three values of ℓ : 17797, 1267 and 12915. These values are taken from Table 4.1 of Hickernell, Hong, L'Ecuyer and Lemieux (2000). We also consider the following two values of (s, m, r) : $(s, m, r) = (3, 4, 4)$ and $(s, m, r) = (2, 5, 5)$. It is expected that both rank-1 lattice rules (6) and (7) will perform well for both of these values of (s, m, r) . It is therefore expected that the new randomization method, described in Section 3, will lead to the estimator $Q'_w f$ of the multiple integral If that has much smaller bias than the estimator $Q_v f$ of this multiple integral that results from the randomization (of the type proposed by Cranley and Patterson, 1976) using iid random variables uniformly distributed on $\{0, 1/2^r, \dots, (2^r - 1)/2^r\}$, described in Section 2. This expectation is borne out by the numerical results presented in Tables 1 and 2.

Bias	$\ell = 17797$	$\ell = 1267$	$\ell = 12915$
$E(Q_v f) - 1$	1.9544×10^{-3}	1.9544×10^{-3}	1.9544×10^{-3}
$E(Q'_w f) - 1$	5.1619×10^{-9}	1.5158×10^{-8}	1.9155×10^{-8}

Table 1: Results for $(s, m, r) = (3, 4, 4)$. Comparison of the bias $E(Q_v f) - 1$ of the estimator that results from the randomization (of the type proposed by Cranley and Patterson, 1976) using sr iid random bits with the bias $E(Q'_w f) - 1$ of the estimator that results from the new randomization, which also uses sr iid random bits.

Bias	$\ell = 17797$	$\ell = 1267$	$\ell = 12915$
$E(Q_v f) - 1$	3.2555×10^{-4}	3.2555×10^{-4}	3.2555×10^{-4}
$E(Q'_w f) - 1$	1.2940×10^{-9}	4.4993×10^{-9}	1.7820×10^{-9}

Table 2: Results for $(s, m, r) = (2, 5, 5)$. Comparison of the bias $E(Q_v f) - 1$ of the estimator that results from the randomization (of the type proposed by Cranley and Patterson, 1976) using sr iid random bits with the bias $E(Q'_w f) - 1$ of the estimator that results from the new randomization, which also uses sr iid random bits.

It is also of interest to compare the standard deviations of the estimators $Q_v f$

and $Q'_w f$. These standard deviations are the square roots of

$$\text{Var}(Q_v f) = \frac{1}{2^{rs}} \sum_{v \in \{0, 1/2^r, \dots, (2^r - 1)/2^r\}^s} (Q_v f - E(Q_v f))^2$$

and

$$\text{Var}(Q'_w f) = \frac{1}{2^{rs}} \sum_{w \in \{0, 1/2^{sr}, \dots, (2^{sr} - 1)/2^{sr}\}} (Q'_w f - E(Q'_w f))^2,$$

respectively. Some numerical values for these standard deviations are presented in Tables 3 and 4. These tables show that, for each of the cases considered, (a) these standard deviations are close and (b) the standard deviation of the estimator $Q_v f$ is comparable to the magnitude of its bias.

Standard deviation	$\ell = 17797$	$\ell = 1267$	$\ell = 12915$
$\sqrt{\text{Var}(Q_v f)}$	7.938×10^{-4}	7.938×10^{-4}	7.938×10^{-4}
$\sqrt{\text{Var}Q'_w f}$	8.389×10^{-4}	8.374×10^{-4}	8.378×10^{-4}

Table 3: Results for $(s, m, r) = (3, 4, 4)$. Comparison of the standard deviation of the estimator $Q_v f$ that results from the randomization (of the type proposed by Cranley and Patterson, 1976) using sr iid random bits with the standard deviation of the estimator $Q'_w f$ that results from the new randomization, which also uses sr iid random bits.

Standard deviation	$\ell = 17797$	$\ell = 1267$	$\ell = 12915$
$\sqrt{\text{Var}(Q_v f)}$	1.6598×10^{-4}	1.6598×10^{-4}	1.6598×10^{-4}
$\sqrt{\text{Var}Q'_w f}$	1.8194×10^{-4}	1.820×10^{-4}	1.782×10^{-4}

Table 4: Results for $(s, m, r) = (2, 5, 5)$. Comparison of the standard deviation of the estimator $Q_v f$ that results from the randomization (of the type proposed by Cranley and Patterson, 1976) using sr iid random bits with the standard deviation of the estimator $Q'_w f$ that results from the new randomization, which also uses sr iid random bits.

5. Conclusion

The new randomization method described in Section 3 requires the extension of a rank-1 lattice rule, for given s , in the number of quadrature points from 2^m to 2^{m+sr} , where sr may be relatively large. Finding such an extension that leads to embedded lattice rules with very good properties for both 2^m quadrature points and 2^{m+sr} quadrature points is clearly an important task.

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Appendix: Review and extension of results for the randomization of Cranley and Patterson (1976) using iid random variables uniformly distributed on $[0, 1]$

Let $y_1 = Q_{\mathbf{u}_1}f, \dots, y_q = Q_{\mathbf{u}_q}f$, where $\mathbf{u}_1, \dots, \mathbf{u}_q$ are independent and identically distributed (iid) uniformly in the unit cube $[0, 1]^s$. Obviously, y_1, \dots, y_q each have the same probability distribution as $y = Q_{\mathbf{u}}f$, where \mathbf{u} is uniformly distributed in the unit cube $[0, 1]^s$. The moments of \bar{y} about its mean can be found from the moments of y , using the well-known properties of cumulants (see e.g. Sections 1.3 and 1.4 of Barndorff-Nielsen and Cox, 1989). The r 'th cumulant $\kappa_r(Y)$ of the random variable Y is defined on p.6 of Barndorff-Nielsen and Cox (1989). This has the following properties. For any number a , $\kappa_r(aY) = a^r \kappa_r(Y)$. If Y_1, \dots, Y_n are iid with the same probability distribution as Y then $\kappa_r(Y_1 + \dots + Y_n) = n \kappa_r(Y)$. Define the r 'th moment about the mean $\mu_r = E[(Y - E(Y))^r]$, $r = 1, 2, \dots$. The μ_r 's can be expressed in terms of the κ_r 's in the way described on p.7 of Barndorff-Nielsen and Cox (1989): $\mu_2 = \kappa_2$, $\mu_3 = \kappa_3$, $\mu_4 = \kappa_4 + 3\kappa_2^2$, etc.. Note that the random variable $y - If$ has bounded support since $|y - If| \leq \sum'_{\mathbf{h} \in L^\perp} |\hat{f}(\mathbf{h})|$. As is well-known, $E(\bar{y} - If) = 0$. This follows immediately from the fact that $E(y - If) = 0$, which can be proved as follows.

$$\begin{aligned} E(y - If) &= E(Q_{\mathbf{u}}f - If) \\ &= E\left(\sum'_{\mathbf{h} \in L^\perp} e^{2\pi i \mathbf{h} \cdot \mathbf{u}} \hat{f}(\mathbf{h})\right) \\ &= \sum'_{\mathbf{h} \in L^\perp} \hat{f}(\mathbf{h}) E(e^{2\pi i h_1 u_1}) \dots E(e^{2\pi i h_s u_s}) \quad \text{by independence of } u_1, \dots, u_s \\ &= 0, \end{aligned}$$

since, for integer h and $u \sim U[0, 1]$,

$$E(e^{2\pi i h u}) = \int_0^1 e^{2\pi i h u} du = \begin{cases} 1 & \text{if } h = 0 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

L'Ecuyer and Lumieux (2000, Proposition 4) (see also L'Ecuyer, Munger and Tuffin, 2010) provide an expression for $E((\bar{y} - If)^2)$ in terms of the Fourier coefficients of f . We may derive this expression as follows. Note that $E((\bar{y} - If)^2) = \text{Var}(\bar{y}) = E((y - If)^2)/q$. Since $\bar{y} - If$ is a real number, it is equal to its complex conjugate, so that

$$\bar{y} - If = \sum'_{\mathbf{k} \in L^\perp} e^{-2\pi i \mathbf{k} \cdot \mathbf{u}} \hat{f}^*(\mathbf{k}),$$

where $\hat{f}^*(\mathbf{k})$ denotes the complex conjugate of $\hat{f}(\mathbf{k})$. Thus

$$\begin{aligned}
E((y - If)^2) &= E\left(\sum'_{\mathbf{h} \in L^\perp} e^{2\pi i \mathbf{h} \cdot \mathbf{u}} \hat{f}(\mathbf{h}) \sum'_{\mathbf{k} \in L^\perp} e^{-2\pi i \mathbf{k} \cdot \mathbf{u}} \hat{f}^*(\mathbf{k})\right) \\
&= \sum'_{\mathbf{h} \in L^\perp} \sum'_{\mathbf{k} \in L^\perp} \hat{f}(\mathbf{h}) \hat{f}^*(\mathbf{k}) E(e^{2\pi i (\mathbf{h} - \mathbf{k}) \cdot \mathbf{u}}) \\
&= \sum'_{\mathbf{h} \in L^\perp} |\hat{f}(\mathbf{h})|^2, \quad \text{by (8).}
\end{aligned}$$

The same kind of argument can be used to find expressions for higher-order moments of \bar{y} , in terms of the Fourier coefficients of f . For example, $E((\bar{y} - If)^3) = E((y - If)^3)/q^2$ and

$$\begin{aligned}
E((y - If)^3) &= E\left(\sum'_{\mathbf{h} \in L^\perp} e^{2\pi i \mathbf{h} \cdot \mathbf{u}} \hat{f}(\mathbf{h}) \sum'_{\mathbf{k} \in L^\perp} e^{-2\pi i \mathbf{k} \cdot \mathbf{u}} \hat{f}^*(\mathbf{k}) \sum'_{\mathbf{l} \in L^\perp} e^{-2\pi i \mathbf{l} \cdot \mathbf{u}} \hat{f}^*(\mathbf{l})\right) \\
&= \sum'_{\mathbf{h} \in L^\perp} \sum'_{\mathbf{k} \in L^\perp} \sum'_{\mathbf{l} \in L^\perp} \hat{f}(\mathbf{h}) \hat{f}^*(\mathbf{k}) \hat{f}^*(\mathbf{l}) E(e^{2\pi i (\mathbf{h} - \mathbf{k} - \mathbf{l}) \cdot \mathbf{u}}) \\
&= \sum_{\mathcal{C}} \hat{f}(\mathbf{h}) \hat{f}^*(\mathbf{k}) \hat{f}^*(\mathbf{l}), \tag{9}
\end{aligned}$$

where $\mathcal{C} = \{(\mathbf{h}, \mathbf{k}, \mathbf{l}) : \mathbf{h} \in L^\perp, \mathbf{h} \neq \mathbf{0}, \mathbf{k} \in L^\perp, \mathbf{k} \neq \mathbf{0}, \mathbf{l} \in L^\perp, \mathbf{l} \neq \mathbf{0}, \mathbf{h} - \mathbf{k} - \mathbf{l} = \mathbf{0}\}$, by (8). Using the fact that a lattice is closed under subtraction, we see that (9) is equal to

$$\sum'_{\mathbf{h} \in L^\perp} \hat{f}(\mathbf{h}) \left(\sum'_{\mathbf{k} \in L^\perp, \mathbf{k} \neq \mathbf{h}} \hat{f}^*(\mathbf{k}) \hat{f}^*(\mathbf{h} - \mathbf{k}) \right)$$